Polytopes I

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1 Monoid Theory

1.1 Group Submonoids

Definition 1.1. A *group subomonoid* is a subset M of a free abelian group $(G, +)$ such that:

i. $\forall m_1, m_2 \in M$, $m_1 + m_2 \in M$ (Closure)

ii. $0_G \in M$ (Identity)

iii. $\forall m_1, m_2 \in M$, $m_1 + m_2 = 0_G \Rightarrow m_1 = m_2 = 0_G$ (Zerosumfree)

To check the definition makes sense we have:

Proposition 1.1. *A group submonoid M of a free abelian group $(G, +)$ is a commutative monoid $(M, +)$.*

Proof. M has closure and identity by definition and inherits associativity and commutativity from $(G, +)$. □

The group submonoid also inherits some of the free structure of G. In the sense that a free abelian group is a module over the ring $\mathbb{Z}$, we have that a group submonoid is a semimodule over the semiring $\mathbb{N}$. Henceforth we refer to group submonoids as simply monoids.

Definition 1.2. The set of *composite elements* of a monoid M is:

$$C(M) := \{m \in M : (\exists \alpha, \beta \in \mathbb{N}, \exists x, y \in M \setminus \{m\} : m = \alpha x + \beta y)\}$$

The *basis* of a monoid M is:

$$B(M) := M \setminus C(M)$$

The *span* $S(X)$ of a subset X of a monoid M is:

$$S(X) := \left\{ \sum_{i=1}^{n} \alpha_i m_i : \alpha_i, n \in \mathbb{N}, m_i \in X \right\}$$

Lemma 1.2. *Suppose M, N are monoids. Then*

$$B(M \times N) = ((B(M) \cup \{0_M\}) \times (B(N) \cup \{0_N\})) \setminus \{0_{M \times N}\}$$

Proof. Suppose $(x, y) \in B(M \times N)$ then $(x, y) \neq (0_M, 0_N)$ and there are no $(m_1, n_1), (m_2, n_2) \in M \times N$ and $\alpha, \beta \in \mathbb{N}$ such that $(x, y) = \alpha(m_1, n_1) + \beta(m_2, n_2)$. Now suppose that $(x, y) \notin ((B(M) \cup \{0_M\}) \times (B(N) \cup \{0_N\})) \setminus \{0_{M \times N}\}$. Since $(x, y) \neq 0_{M \times N}$, $x \notin B(M) \cup \{0_M\}$ and $y \notin B(N) \cup \{0_N\}$. Hence $\exists \alpha_x, \alpha_y, \beta_x, \beta_y \in \mathbb{N}, m_1, m_2 \in M, n_1, n_2 \in N$ such that: $x = \alpha_x m_1 + \beta_x m_2$ and $y = \alpha_y n_1 + \beta_y n_2$. Combining these equations gives:

$$\begin{aligned} (x, y) &= (\alpha_x m_1 + \beta_x m_2, \alpha_y n_1 + \beta_y n_2) \\ &= (\alpha_x m_1, \alpha_y n_1) + (\beta_x m_2, \beta_y n_2) \end{aligned}$$

Since $(\alpha_x m_1, \alpha_y n_1), (\beta_x m_2, \beta_y n_2) \in M \times N$, we have a contradiction and so $x \in ((B(M) \cup \{0_M\}) \times (B(N) \cup \{0_N\})) \setminus \{0_{M \times N}\}$. Hence $B(M \times N) \subseteq ((B(M) \cup \{0_M\}) \times (B(N) \cup \{0_N\})) \setminus \{0_{M \times N}\}$.

Now suppose $x \in ((B(M) \cup \{0_M\}) \times (B(N) \cup \{0_N\})) \setminus \{0_{M \times N}\}$. Then □

Theorem 1.3. *$B(M)$ is the smallest subset of M such that $S(B(M)) = M$.*

Proof. We first show that $S(B(M)) = M$. Clearly $S(B(M)) \subseteq M$. Now suppose $m \in M$. If $m \in B(M)$ then $m \in S(B(M))$. So suppose that $m \notin B(M)$. Therefore $m \in C(M)$ and thus $\exists \alpha, \beta \in \mathbb{N}, \exists x, y \in M : m = \alpha x + \beta y$. If such $x, y \in B(M)$ then $m \in S(B(M))$ and we are finished. If not, we can break down the x or y and repeat the process until we find a contruction of m using a linear combination of elements from $B(M)$. Therefore $m \in S(B(M))$ and so $M \subseteq S(B(M))$.

Now suppose that there was a set smaller than $B(M)$ with this property and call it X. Choose $x \in B(M) \setminus X$. Since $M = S(X)$ there is a linear combination of elements of X equal to x. But since $x \in B(M)$ this is a contradiction. So $B(M)$ is the smallest set. □

Definition 1.3. The *rank* of a monoid M is given by $|B(M)|$.

2 Introduction to Polytopes

2.1 Convex Hulls

Definition 2.1. Given $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^n$, define a *convex linear combination* of $\mathbf{x}_1, \ldots \mathbf{x}_n$ as:

$$\alpha_1 \mathbf{x}_1 + \ldots + \alpha_n \mathbf{x}_n$$

with $\alpha_i \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

Definition 2.2. The *finite convex hull* of a set of points $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^n$ is the set of all convex combinations of the points. i.e.

$$\text{conv}^*(\mathbf{x}_1, \ldots \mathbf{x}_n) := \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i : \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

Definition 2.3. A *hyperplane* $H(\mathbf{h}, \alpha) \subseteq \mathbb{R}^n$ is a set of the form:

$$H(\mathbf{h}, \alpha) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}.\mathbf{h} = \alpha\}$$

We say that $H(\mathbf{h}, \alpha)$ *supports* a subset $A \in \mathbb{R}^n$ if $\forall \mathbf{x} \in A$, either $\mathbf{x}.\mathbf{h} \geq \alpha$ or $\mathbf{x}.\mathbf{h} \leq \alpha$.

Definition 2.4. A *convex polytope* $P \in \mathcal{P}(\mathbb{R}^n)$ is a convex hull such that for every hyperplane $H \subseteq \mathbb{R}^n$, $P \nsubseteq H$.

We call the set of convex polytopes $\mathbb{R}^n$ *the well* and denote it $\mathbb{W}^n \subseteq \mathcal{P}(\mathbb{R}^n)$.

Definition 2.5. Suppose $P \subseteq \mathbb{R}^n$. A point $p \in P$ is an *extreme point* of P if $\forall X \subseteq P$ such that $|X| < \infty$, $p \notin \text{conv}^*(X)$ and we denote the set of extreme points of $\text{ext}(P)$.

Theorem 2.1. $\text{ext}(X)$ *is the smallest subset of* $X \subseteq \mathbb{R}^n$ *such that* $\text{conv}^*(\text{ext}(X)) = \text{conv}^*(X)$.

Definition 2.6. Given a subset $X \subseteq \mathbb{R}^n$, we define the *convex hull* of X as $\text{conv}(X) = \text{conv}^*(\text{ext}(X))$.

2.2 Faces

Definition 2.7. A subset F of a set $P \subseteq \mathbb{R}^n$ is called a *face* of P if there exists a supporting hyperplane $H \subseteq \mathbb{R}^n$ such that $F = H \cap P$, $F = P$ or $F = \varnothing$.

Definition 2.8. The *convex face function* $\mathcal{F}^* : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$ is defined as:

$$\mathcal{F}^*(P) := \{F \subseteq P : F \text{ is a face of } P\}$$

Face here is used as a general term. For example for $P \in \mathbb{W}^3$, $\mathcal{F}^*(P)$ includes all of the vertices, edges and faces of P. For non-convex objects, $\mathcal{F}^*$ fails to capture those subsets that are "concave" and so we require a more general notion of a face.

Definition 2.9. The *closure-boundary operator* $\mathcal{N} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is defined as:

$$\mathcal{N}(P) := \overline{P} \setminus P$$

Definition 2.10. The *face function* $\mathcal{F}^* : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$ is defined as:

$$\mathcal{F}(P) := \mathcal{F}^*(P) \cup \mathcal{F}^*(\mathcal{N}(\mathrm{conv}(P) \setminus P))$$

Proposition 2.2. *Suppose $P \in \mathbb{W}^n$. Then $\mathcal{F}(P) = \mathcal{F}^*(P)$.*

For $X \subseteq \mathbb{R}^n$, $\overline{X}$ denotes the *closure* of X.

Proof. Suppose $P \in \mathbb{W}^n$. Then:

$$\begin{aligned}
\mathcal{F}(P) &= \mathcal{F}^*(P) \cup \mathcal{F}^*(\mathcal{N}(\mathrm{conv}(P) \setminus P)) \\
&= \mathcal{F}^*(P) \cup \mathcal{F}^* \left(\overline{(\mathrm{conv}(P) \setminus P)} \setminus (\mathrm{conv}(P) \setminus P)\right) \\
&= \mathcal{F}^*(P) \cup \mathcal{F}^* \left(\overline{(P \setminus P)} \setminus (P \setminus P)\right) \\
&= \mathcal{F}^*(P) \cup \mathcal{F}^* (\overline{\varnothing} \setminus \varnothing) \\
&= \mathcal{F}^*(P) \cup \mathcal{F}^* (\varnothing \setminus \varnothing) \\
&= \mathcal{F}^*(P) \cup \mathcal{F}^* (\varnothing) \\
&= \mathcal{F}^*(P) \cup \varnothing \\
&= \mathcal{F}^*(P)
\end{aligned}$$

□

Observe that the collection $\mathcal{F}(P)$ is partially ordered by inclusion and, in the case of convex polytopes, a lattice. We can use this partial ordering to define the dimension of elements of the collection.

Definition 2.11. Suppose P is a polytope and $F \in \mathcal{F}(P)$. We define the dimension of F, $\dim F$, as the number of subset inclusions F has in $\mathcal{F}(P)$.

Again, for example: any planar face has dimension 2 since it has two inclusions: edges and vertices contained in those edges. Similary a polyhedron has dimension 3. Notice that we can partition $\mathcal{F}(P)$ by the dimension of its elements.

Definition 2.12. We define the dth face subcollection of a polytope P as:

$$\mathcal{F}_d(P) := \{F \in \mathcal{F}(P) : \dim F = d\}$$

Proposition 2.3. *The face subcollections of a polytope P partition $\mathcal{F}(P)$.*

Proof. Suppose $F \in \mathcal{F}_d(P)$ then $\dim F = d$ and so $\forall n \in \mathbb{N} \setminus \{d\}$, $F \notin \mathcal{F}_n(P)$. Hence the subcollections are disjoint. Clearly every face has a dimension and so every face must be contained in $\mathcal{F}_d(P)$ for some $d \in \mathbb{N}$. Hence $\mathcal{F}(P) = \bigcup_{i=0}^{n} \mathcal{F}_i(P)$. □

2.3 Polytopic Equivalence

Now that we are equipped with a sufficient notion of what convex polytopes are, we can attempt to remove some geometric structure and focus on the combinatorial properties.

Definition 2.13. Suppose $P, Q \in \mathbb{W}^n$. We say P and Q are *combinatorially equivalent* or *c-equivalent* and write $P \cong Q$ if there is a bijection $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ such that $\forall F, G \in \mathcal{F}(P)$, $F \subseteq G \Leftrightarrow \phi(F) \subseteq \phi(G)$.

Equivalently, $P \cong Q$ if and only if $\mathcal{F}(P) \cong \mathcal{F}(Q)$. That is, $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ are isomorphic as partially ordered sets.

Proposition 2.4. *C-equivalence is an equivalence relation.*

Proof. Clearly $P \cong P$ via the identity map. Suppose $P \cong Q$ with isomorphsim ϕ. Then $\phi^{-1} : \mathcal{F}(Q) \to \mathcal{F}(P)$ is a bijection and $\forall \phi(F), \phi(G) \in \mathcal{F}(Q)$, $\phi(F)) \subseteq \phi(G) \Leftrightarrow F \subseteq G \Leftrightarrow \phi^{-1}(\phi(F)) \subseteq \phi^{-1}(\phi(G))$. Hence $Q \cong P$. Similary if $Q \cong P$ then $P \cong Q$. Finally suppose $P, Q, R \in \mathbb{W}^n$ and $P \cong Q$, $Q \cong R$ with isomorphisms ϕ, ψ respectively. $\psi \circ \phi$ is a bijection and $\forall F, G \in \mathcal{F}(P)$, $F \subseteq G \Leftrightarrow \phi(F) \subseteq \phi(G) \Leftrightarrow \psi(\phi(F)) \subseteq \psi(\phi(G))$. Hence $P \cong R$. ☐

This is the natural notion of combinatorial equivalence as it preseves only the inherent structure of the shape and not how it is stretched or skewed or how regular it is etc. However it is slightly too strict an equivalence for what we require as is illustrated in the follwing example.

Proposition 2.5. *The dodecahedron is not c-equivalent to the decagonal prism.*

Proof. Choose any 2-face of the dodecahedron and call it F_2. F_2 must either map to a decagonal face of the prism or a rectangular one. Suppose it maps to the latter. Then each edge subset of F_2 must map to an edge of the rectangular face. This is impossible since we are mapping 4 edges to 5. Hence F_2 must map to the decagonal face. This can be done for at most two 2-faces of the dodecahedron and since we have 12 to map, one must map to a rectangular face. Therefore, any bijection we choose cannot be inclusion preserving. Thus, the two shaped are not c-equivalent. ☐

This example illustrates the defficiency in c-equivalence. Both shapes have the same number of vertices, edges and faces[1] but are not equivalent. Hence we wish to define an equivalence that preserves the number of each type of face rather than inclusion.

Definition 2.14. Suppose $P, Q \in \mathbb{W}^n$. We say P and Q are *B-equivalent* and write $P \sim_B Q$ if there is a bijection $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ such that $\forall d \in \mathbb{N}$, $|\mathcal{F}_d(P)| = |\phi(\mathcal{F}_d(P))|$.

Proposition 2.6. *B-equivalence is an equivalence relation.*

Proof. Clearly $P \sim_B P$ via the identity map. Suppose $P \sim_B Q$ with isomorphsim ϕ. Then $\phi^{-1} : \mathcal{F}(Q) \to \mathcal{F}(P)$ is a bijection and $\forall d \in \mathbb{N}$, $|\phi(\mathcal{F}_d(P))| = |\mathcal{F}_d(P)| = |\phi^{-1}(\phi(\mathcal{F}_d(P)))|$. Hence $Q \sim_B P$. Similary if $Q \sim_B P$ then $P \sim_B Q$. Finally suppose $P, Q, R \in \mathbb{W}^n$ and $P \sim_B Q$, $Q \sim_B R$ with isomorphisms ϕ, ψ respectively. $\psi \circ \phi$ is a bijection and $\forall d \in \mathbb{N}$, $|\mathcal{F}_d(P)| = |\phi(\mathcal{F}_d(P))| = |\psi(\phi(\mathcal{F}_d(P)))|$ Hence $P \sim_B R$. ☐

Cleary this definition sets any two polytopes with the same number of d-faces[2] as equivalent, removing the issue displayed in proposition 2.5. The two notions of equivalence are related in the following way.

Proposition 2.7. *Suppose $P, Q \in \mathbb{W}^n$. Then $P \cong Q \Rightarrow P \sim_B Q$.*

[1] $20, 30, 12$.
[2] Face of dimension d.

Proof. Suppose $P \cong Q$ and $\phi : \mathcal{F}(P) \to \mathcal{F}(Q)$ is the inclusion preserving one-to-one map. Since the dimension of a face depends only on set inclusion, we have that $\forall F \in \mathcal{F}(P)$, $\dim F = \dim \phi(F)$ and so $F \in \mathcal{F}_d(P) \Leftrightarrow \phi(F) \in \phi(\mathcal{F}_d(P))$. Hence $\forall d \in \mathbb{N}$, $|\mathcal{F}_d(P)| = |\phi(\mathcal{F}_d(P))|$. □

Note that propostion 2.5 shows that $P \sim_B Q \nRightarrow P \cong Q$ and thus we know that $P \cong Q \nLeftrightarrow P \sim_B Q$.

2.3.1 Spherical Polytopes

We now extend the convex polytopes to include all polytopes that are combinatorially equivalent to convex polytopes.

Definition 2.15. The set of *spherical polytopes* is defined as:

$$\mathbb{Y}_0^n := \{P \subseteq \mathbb{R}^n : \exists Q \in \mathbb{W}^n \text{ such that } P \cong Q\}$$

We further remove geometric stucture from $\mathbb{Y}_0^n$ by quotienting the set by B-equivalence and treating spherical polytopes as elements of the set:

$$\mathbb{Y}_0^n / \sim_B := \{[P]_{\sim_B} : P \in \mathbb{Y}_0^n\}$$

2.4 The Natural Numbers

We now switch our representation of spherical polytopes to elements of $\mathbb{N}^n$.

Definition 2.16. Define the map $E : \mathbb{Y}_0^n / \sim_B \to \mathbb{N}^n$ as:

$$E([P]_{\sim_B}) := \begin{pmatrix} |\mathcal{F}_0(P)| \\ \vdots \\ |\mathcal{F}_{n-1}(P)| \end{pmatrix}$$

We first show that E is well-defined and injective

Proposition 2.8. *Suppose $P, Q \in \mathbb{F}_0^n$. Then $P \sim_B Q$ if and only if $E([P]) = E([Q])$.*

Proof. Suppose $P, Q \in \mathbb{F}_0^n$ such that $P \sim_B Q$. Hence $\forall d \in \mathbb{N}$, $|\mathcal{F}_d(P)| = |\phi(\mathcal{F}_d(P))| = |\mathcal{F}_d(Q)|$. Therefore $E([P]_{\sim_B}) = E([Q]_{\sim_B})$. Now suppose that $E([P]_{\sim_B}) = E([Q]_{\sim_B})$. Then $\forall d \in \mathbb{N}$, $|\mathcal{F}_d(P)| = |\phi(\mathcal{F}_d(P))| = |\mathcal{F}_d(Q)|$ and hence $P \sim_B Q$. Thus, $[P]_{\sim_B} = [Q]_{\sim_B}$ and so E is injective. □

We denote the set of tupels representing spherical polytopes by $\mathbb{F}_0^n := E(\mathbb{Y}_0^n / \sim_B)$ so that E is a bijection between $\mathbb{Y}_0^n / \sim_B$ and $\mathbb{F}_0^n$.

3 Structure of $\mathbb{F}_0^n$

3.1 Simplices

Definition 3.1. The n-dimensional simplex $a_n \in \mathbb{F}_0^n$ is given by:

$$a_n := \begin{pmatrix} \binom{n+1}{1} \\ \vdots \\ \binom{n+1}{n} \end{pmatrix}$$

Additionally we allow lower dimensional simplices $a_m \in \mathbb{N}^n$ with $m < n$ to interact with elements of $\mathbb{F}_0^n$ by defining:

$$a_m := \begin{pmatrix} \binom{m+1}{1} \\ \vdots \\ \binom{m+1}{n} \end{pmatrix}$$

Example 3.1. Some familiar examples in $\mathbb{F}^3$:

i. Triangle $a_2 := \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

ii. Tetrahedron $a_3 := \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix}$

3.1.1 Matrix Functions

Before we move onto the next section, we require the following very simple notion allowing functions to act on tupels.

Definition 3.2. Suppose $X \subseteq \mathbb{R}$. Given a function $f : \{0, \dots, n\} \to X$, we define the matrix function G of f as:

$$G(f) := \begin{pmatrix} f(0) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f(n) \end{pmatrix}$$

3.2 Polytopic Simplical Constructions

Consider 'gluing' a simpliex to another of the same dimension via a common facet. Observe that the resultant shape is indeed a polytope with simplicial facets. To construct this face we would identify the lower dimensional faces appropriatley via an order homomorphism and then remove the facets that were identified together since they are now on the 'inside' of the polytope. Using this construction we further oberve that the tupel of this new shape is formed as follows:

$$z = a_n + (a_n - a_{n-1} - e_n)$$

where e_n is the nth standard basis vector of $\mathbb{R}^n$. We can continue this process for as long as we like, say for $t \in \mathbb{N}$ iterations, each time gluing a new simplex to the polytope. Hence we have constructed a family of polytopes in $\mathbb{F}_0^n$ of the form:

$$z = a_n + t(a_n - a_{n-1} - e_n)$$

A natural next question to ask is whether a shape like a cube is a memer of this family. Clearly the cube in $\mathbb{Y}_0^n$ itself is not but there may be a member of this family B-equivalent to it.

Proposition 3.1. *$\nexists t \in \mathbb{N}$ such that:*

$$\begin{pmatrix} 8 \\ 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} + t \left(\begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Proof. We simplify the equation to get:

$$\begin{pmatrix} 8 \\ 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

from which we deduce that $t = 4 \wedge t = 2 \wedge t = 1$. Hence there is no single $t \in \mathbb{N}$ generating the cube. $\square$

As trivial as this proposition is, it highlights an interesting pattern. If we replace the constant t with a function $f : \{0, 1, 2\} \to \mathbb{Q}_{\geq 0}$ defined by $f(0) = 4, f(1) = 2, f(2) = 1$ we can generate the cube from this construction. Admittedly, we have lost the ability to visualise what is going on in this construction but there is something very 'cube-like' about the above function.

Evidently every element of $\mathbb{N}^n$ can be written in the form:

$$z = a_n + G(f)(a_n - a_{n-1} - e_n)$$

for some function $f : \mathbb{N} \to \mathbb{Q}_{\geq 0}$ and hence every element of $\mathbb{F}_0^n$ can be written in the form:

$$z = a_n + G(f)(a_n - a_{n-1} - e_n)$$

for some function $f : \mathbb{N} \to \mathbb{Q}_{\geq 0}$. The exact restrictions on what functions can be chosen is an equivalent problem to the classification of the tupels of convex polytopes. We will address this problem for $\mathbb{F}^3$ and below using *Steinitz's theorem* later. For now we move onto the most important conjecture that will be presented here. It concerns the one of the ultimite goals of this study. That is, to define a notion of combination or 'addition' of polytopes.

3.3 Polytopic Addition

Definition 3.3. Suppose $z \in \mathbb{F}_0^n$. Then we define the shape function of z, $g_z : \{0, \ldots, n\} \to \mathbb{Q}_{\geq 0}$ as the solution to the equation:

$$z = a_n + G(g_z)(a_n - a_{n-1} - e_n)$$

Conjecture 3.2. *Suppose $p, q \in \mathbb{F}_0^n$. Then $\exists r \in \mathbb{F}_0^n$ such that:*

$$r = a_n + (G(g_p) + G(g_q))(a_n - a_{n-1} - e_n)$$

Assuming this conjecture holds, we can then define a notion of addition in $\mathbb{F}_0^n$.

Definition 3.4. We define addition between elements $p, q \in \mathbb{F}_0^n$ as:

$$p +_c q := a_n + (G(g_p) + G(g_q))(a_n - a_{n-1} - e_n)$$

Proposition 3.3. $\forall p, q \in \mathbb{F}_0^n$,

$$p +_c q = p + q - a_n$$

Proof.

$$\begin{aligned} p +_c q &= a_n + (G(g_p) + G(g_q))(a_n - a_{n-1} - e_n) \\ &= a_n + G(g_p)(a_n - a_{n-1} - e_n) + G(g_q)(a_n - a_{n-1} - e_n) \\ &= a_n + G(g_p)(a_n - a_{n-1} - e_n) + a_n + G(g_q)(a_n - a_{n-1} - e_n) - a_n \\ &= p + q - a_n \end{aligned}$$

□

Proposition 3.4. $\forall n \in \mathbb{N} \setminus \{0\}$,

$$a_n - a_{n-1} - e_n = \begin{pmatrix} \binom{n}{0} \\ \vdots \\ \binom{n}{n-2} \\ \binom{n}{n-1} - 1 \end{pmatrix}$$

Proof. Simple after application of the Pascal's triangle lemma for binomial coefficients. □

Proposition 3.5. $\forall z \in \mathbb{F}_0^n$,

$$g_z(t) = \begin{cases} \frac{z.e_{t+1}}{\binom{n}{t}} - \frac{n+1}{t+1} & 0 \leq t < n-1 \\ \frac{z.e_n}{n-1} - \frac{n+1}{n-1} & t = n-1 \end{cases}$$

Proof. Follows from proposition 3.4. □

4 Low Dimensional Polytopes

4.1 Dimensions 0, 1 and 2

Observe that in $\mathbb{Y}_0^0$, there is only one polytope - the single point. Hence we conclude that $\mathbb{F}_0^0 = \{a_0\}$. Now suppose we take the convex hull of a collection of points in $\mathbb{R}$. The result is either a member of $\mathbb{F}_0^0$ or is a line segment with two end point vertices. Hence $\mathbb{F}_0/\sim_B$ contains only one equivalence class. Therefore: $\mathbb{F}_0 = \{2\}$. The equivalence classes of the polytopes of $\mathbb{F}_0^2$ are given by the number of vertices of the polytope since each polytope has number of vertices and edges equal. However when there are less than three vertices, the polytopes are points and line segments not in $\mathbb{Y}_0^2$. Hence we have that:

$$\mathbb{F}_0^2 = \left\{ \begin{pmatrix} n \\ n \end{pmatrix} : (n \in \mathbb{N} : n \geq 3) \right\}$$

Proposition 4.1. $(\mathbb{F}_0^0, +_c)$ *is a commutative monoid.*

Proof. We make use of proposition 3.3. $a_0 +_c a_0 = a_0 + a_0 - a_0 = a_0$. Hence a_0 is the idensity element and $\mathbb{F}_0^0$ is closed under the C-addition. C-addition is also commutative and associative on $\mathbb{N}^n$ by proposition 3.3. □

Proposition 4.2. $(\mathbb{F}_0, +_c)$ *is a commutative monoid.*

Proof. The proof is identical to the case for dimension 0 as $\mathbb{F}_0 = \{a_1\}$. □

Proposition 4.3. $(\mathbb{F}_0^2, +_c)$ *is a commutative monoid.*

Proof. We just have to show identity and closure. Observe that a_n is always the identity element of C-addition. Now suppose that:

$$\begin{pmatrix} n \\ n \end{pmatrix}, \begin{pmatrix} m \\ m \end{pmatrix} \in \mathbb{F}_0^2$$

Then since $n, m \geq 3$,

$$\begin{pmatrix} n \\ n \end{pmatrix} +_c \begin{pmatrix} m \\ m \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix} + \begin{pmatrix} m \\ m \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} n+m-3 \\ n+m-3 \end{pmatrix} \in \mathbb{F}_0^2$$

□

4.2 Dimension 3

We now make concrete some of the simplicial contruction theory from the previois section for dimensions three.

Definition 4.1. The *second Euler charectaristic* $\chi_2 : \mathbb{F}_0^n \to \mathbb{Z}$ is defined as:

$$\chi_2(z) := -1 + \sum_{i=0}^{n-1} (-1)^i z.e_{i+1} + (-1)^n$$

Observe that the second Euler characteristic is related to the *Euler characteristic* in the following way: $\chi_2(z) = -1 + \chi(z) + (-1)^n$.

Theorem 4.4. (Steinitz's Theorem) *Suppose $z \in \mathbb{N}^n$. Then if:*

i. $4 \leq \frac{z.e_3+4}{2} \leq z.e_1 \leq 2z.e_3 - 4$

ii. $\chi_2(z) = 0$

$$g_z(t) = \begin{cases} z.e_1 - 4 & t = 0 \\ \frac{z.e_2}{3} - 2 & t = 1 \\ \frac{z.e_3}{2} - 2 & t = 2 \end{cases}$$

Corollary. (Steinitz's Theorem II) *Suppose $z \in \mathbb{N}^n$. Then if:*

i. $0 \leq g_z(2) \leq g_z(0) \leq 4g_z(2)$

ii. $\chi_2(z) = 0$

5 Other

Definition 5.1. A function $g : \{0, 1, 2\} \to \mathbb{Q}_{\geq 0}$ is *Grünbaum* if:

i. $g(0) \leq 2g(1) \leq 4g(2) \leq 4g(0)$

ii. $g(0) - 3g(1) + 2g(2) = 0$

We denote the set of Grünbaum functions on $\{0, 1, 2\}$ by $\mathcal{G}^3$.

Proposition 5.1. *$\mathcal{G}^3$ is closed under additon.*

Proof. Suppose $g_1, g_2 \in \mathcal{G}^3$. Observe that $g_1 + g_2$ satisfies i. It remains to check ii. Consider:

$$\begin{aligned} (g_1 + g_2)(0) - 3(g_1 + g_2)(1) + 2(g_1 + g_2)(2) &= g_1(0) + g_2(0) - 3(g_1(1) + g_2(1)) + 2(g_1(2) + g_2(2)) \\ &= g_1(0) + g_2(0) - 3g_1(1) - 3g_2(1) + 2g_1(2) + 2g_2(2) \\ &= g_1(0) - 3g_1(1) + 2g_1(2) + g_2(0) - 3g_2(1) + 2g_2(2) \\ &= 0 \end{aligned}$$

Hence $g_1 + g_2 \in \mathcal{G}^3$. □

Definition 5.2. The *Grünbaum* matrix $G \in \mathbb{Q}_{\geq 0}^{3\times 3}$ of a function $g \in \mathcal{G}^3$ is:

$$G := \begin{pmatrix} g(0) & 0 & 0 \\ 0 & g(1) & 0 \\ 0 & 0 & g(2) \end{pmatrix}$$

Proposition 5.2. *The sum of two Grünbaum matrices is a Grünbaum matrix.*

Proof. Follows from proposition 5.1. □

Definition 5.3. The *tetrahedral construction* matrix is:

$$A_3 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Definition 5.4. The set of genus 0 polyhedra is:

$$\mathbb{F}^3 := \{z_3 \in \mathbb{N}^4 : z_3 = a_3 + G_z A_3 a_2\}$$

Where G_z is the Grünbaum matrix corresponding to z.

Example 5.1. For lower dimensions we have:

i. $\mathbb{F}^0 = \{a_0\}$

ii. $\mathbb{F}^1 = \{a_1\}$

Definition 5.5. We define addition between elements $x, y \in \mathbb{F}^3$ as:

$$x +_c y := a_M + (G_x + G_y)A_3 a_2$$

Proposition 5.3. *llx*, $y \in \mathbb{F}^3$,

$$x +_c y = x + y - a_3$$

Proof.

$$\begin{aligned} x + y - a_3 &= a_M + G_x A_3 a_2 + a_3 + G_y A_3 a_2 - a_3 \\ &= a_3 + G_x A_3 a_2 + G_y A_3 a_2 \\ &= a_M + (G_x + G_y)A_3 a_2 \\ &= x +_c y \end{aligned}$$

□

Proposition 5.4. $(\mathbb{F}^3, +_c)$ *is a monoid.*

Proof. a_3 is the identity element and $\mathbb{F}^3$ is closed under addition by propositions 5.2 and 3.3. □

Theorem 5.5. (Euler's Formula for Polyhedra) $\forall z \in \mathbb{F}^3$, $(1, -1, 1, -1)z = 1$.

Proof.

$$\begin{aligned} (1,-1,1,-1)z &= (1,-1,1,-1)(a_3 + G_z A_3 a_2) \\ &= (1,-1,1,-1)\left(\begin{pmatrix}4\\6\\4\\1\end{pmatrix} + \begin{pmatrix} g_z(0) & 0 & 0 \\ 0 & g_z(1) & 0 \\ 0 & 0 & g_z(2)\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}3\\3\\1\end{pmatrix}\right) \\ &= (1,-1,1,-1)\left(\begin{pmatrix}4\\6\\4\\1\end{pmatrix} + \begin{pmatrix} 0 & 0 & g_z(0) \\ g_z(1) & 0 & 0 \\ 0 & g_z(2) & -g_z(2)\end{pmatrix}\begin{pmatrix}3\\3\\1\end{pmatrix}\right) \\ &= (1,-1,1,-1)\left(\begin{pmatrix}4\\6\\4\\1\end{pmatrix} + \begin{pmatrix} g_z(0) \\ 3g_z(1) \\ 2g_z(2)\end{pmatrix}\right) \\ &= (1,-1,1,-1)\begin{pmatrix}4 + g_z(0)\\ 6 + 3g_z(1)\\ 4 + 2g_z(2)\\ 1\end{pmatrix} \\ &= 4 + g_z(0) - 6 - 3g_z(1) + 4 + 2g_z(2) - 1 \\ &= 1 + g_z(0) - 3g_z(1) + 2g_z(2) \\ &= 1 \end{aligned}$$

□

5.1 Polytopic Simplicial Constructions

We can generalise the definition of $\mathbb{F}^3$ fairly easily by changing 3 to $M \in \mathbb{N}$ in the definitions and expanding the two matrices accordingly. The difficulty in the generalisation arises from $\mathcal{G}^M$, that is the set of all Grünbaum functions on $\{0, 1, \ldots M - 1\}$. We require a definition of Grünbaum functions on sets with cardinality greater than 3.

5.2 Minkowski Addition

We move now to define addition between polytopes in order to give $\mathbb{Y}^n$ some kind of algebraic structure.

Definition 5.6. Suppose $P, Q \in \mathbb{Y}^n$ then we define:

$$P + Q := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in P, \mathbf{q} \in Q\}$$

Lemma 5.6. *Suppose $A, B \subseteq \mathbb{R}^n$ then $\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B)$.*

Proof.

$$\begin{aligned}
\mathrm{conv}(A+B) &= \mathrm{conv}(\{\mathbf{a}+\mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}) \\
&= \mathrm{conv}(\mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n) \qquad \mathbf{a}_i \in A, \mathbf{b}_i \in B \\
&= \left\{ \sum_{i=1}^{n} \alpha_i(\mathbf{a}_i + \mathbf{b}_i) : \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} \alpha_i = 1 \right\} \\
&= \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{a}_i + \sum_{i=1}^{n} \alpha_i \mathbf{b}_i : \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} \alpha_i = 1 \right\} \\
&\subseteq \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{a}_i : \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} \alpha_i = 1 \right\} + \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{b}_i : \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} \alpha_i = 1 \right\} \\
&= \mathrm{conv}(A) + \mathrm{conv}(B)
\end{aligned}$$

So $\mathrm{conv}(A + B) \subseteq \mathrm{conv}(A) + \mathrm{conv}(B)$. Now suppose $\mathbf{v} + \mathbf{w} \in \mathrm{conv}(A) + \mathrm{conv}(B)$ with $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i$ and $\mathbf{w} = \sum_{i=1}^{n} \beta_i \mathbf{b}_i$. Consider:

$$\begin{aligned}
\mathbf{v} + \mathbf{b}_j &= \sum_{i=1}^{n} \alpha_i \mathbf{a}_i + \mathbf{b}_j \sum_{i=1}^{n} \alpha_i \\
&= \sum_{i=1}^{n} \alpha_i (\mathbf{a}_i + \mathbf{b}_j)
\end{aligned}$$

So $\mathbf{v} + \mathbf{b}_j \in \mathrm{conv}(A + B)$. Now consider:

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= \mathbf{v} \sum_{j=1}^{n} \beta_j + \mathbf{b}_j \sum_{j=1}^{n} \beta_i \mathbf{b}_j \\
&= \sum_{j=1}^{n} \beta_j (\mathbf{v} + \mathbf{b}_j)
\end{aligned}$$

Hence $\mathbf{v} + \mathbf{w} \in \mathrm{conv}(\mathrm{conv}(A + B)) = \mathrm{conv}(A + B)$. Therefore $\mathrm{conv}(A) + \mathrm{conv}(B) \subseteq \mathrm{conv}(A + B)$. □

Lemma 5.7. *Suppose $P, Q \subseteq \mathbb{R}^n$ and that for every hyperplane $H \subseteq \mathbb{R}^n$, $P, Q \nsubseteq H$. Then for every hyperplane $H \subseteq \mathbb{R}^n$, $P + Q \nsubseteq H$.*

Proof. Since for every hyperplane $H \subseteq \mathbb{R}^n$, $P, Q \nsubseteq H$, we have that there are no $\beta, \gamma \in \mathbb{R}$ and $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ such that $\forall \mathbf{p} \in P, \mathbf{q} \in Q$,

$$\mathbf{p}.\mathbf{f} = \beta \qquad\qquad \mathbf{q}.\mathbf{g} = \gamma$$

Hence there are no $\beta, \gamma \in \mathbb{R}$ and $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ such that $\forall \mathbf{p} \in P, \mathbf{q} \in Q$,

$$\mathbf{p}.\mathbf{f} + \mathbf{q}.\mathbf{g} = \beta + \gamma$$

Therefore there is no $\alpha \in \mathbb{R}$ and $\mathbf{h} \in \mathbb{R}^n$ such that $\forall \mathbf{p} \in P, \mathbf{q} \in Q$,

$$\mathbf{p}.\mathbf{h} + \mathbf{q}.\mathbf{h} = \alpha$$

Thus, there is no $\alpha \in \mathbb{R}$ and $\mathbf{h} \in \mathbb{R}^n$ such that $\forall \mathbf{p} \in P, \mathbf{q} \in Q$,

$$(\mathbf{p} + \mathbf{q}).\mathbf{h} = \alpha$$

Finally we have that there is no $\alpha \in \mathbb{R}$ and $\mathbf{h} \in \mathbb{R}^n$ such that $\forall \mathbf{p} + \mathbf{q} \in P + Q$,

$$(\mathbf{p} + \mathbf{q}).\mathbf{h} = \alpha$$

So there is no hyperplane $H \subseteq \mathbb{R}^n$ such that $P + Q \subseteq H$. □

Theorem 5.8. *Suppose $P, Q \in \mathbb{Y}^n$ then $P + Q \in \mathbb{Y}^n$.*

Proof. Consider:

$$\begin{aligned} P + Q &= \operatorname{conv}(P) + \operatorname{conv}(Q) \\ &= \operatorname{conv}(P + Q) && \text{by lemma 5.6} \end{aligned}$$

Also, since $P, Q \in \mathbb{Y}^n$, there is no hyperplane $H \subseteq \mathbb{R}^n$ containing them and hence by lemma 5.7, there is no hyperplane $H \subseteq \mathbb{R}^n$ containing $P + Q$. Therefore $P + Q \in \mathbb{Y}^n$. □

Theorem 5.9. $(\mathbb{Y}^n, +)$ *is a commutative monoid.*

Proof. Closure follows from theorem 5.8. Associativity and commutativity follow from the fact that vector addition in $\mathbb{R}^n$ is associative and commutative. Finally, the identity polytope $\{\mathbf{0}\}$ is the identity element since $\forall P \in \mathbb{Y}^n$, $P + \{\mathbf{0}\} = \{\mathbf{p} + \mathbf{0} : \mathbf{p} \in P\} = P$. □

Note that by lemma 5.7 it is also zerosumfree. We would like to examine the effect of Minkowski addition on the face collections of polytopes.

Definition 5.7. Suppose $X \subseteq \mathbb{R}^n$ and $x \in X$. x is an *extreme point* of X if $\operatorname{conv}(X \setminus \{x\}) \neq \operatorname{conv}(X)$.

Observe that if $X \in \mathbb{Y}^n$ then the set of extreme points in X is $\bigcup_{F \in \mathcal{F}_0(X)} F$. In other words, the vertices of X. We can use this to write every $P \in \mathbb{Y}^n$ as $P = \operatorname{conv}\left(\bigcup_{F \in \mathcal{F}_0(P)} F\right)$.